JOURNAL OF COMBINATORIAL THEORY, Series A 59, 177–186 (1992)

Gray Arrays and Gray Tori

L. J. CUMMINGS

University of Waterloo, Waterloo, Ontario, Canada N2L 3G1

Communicated by Andrew Odlyzko

Received September 12, 1989

A Gray array is an $m \times n$ array of elements chosen from a finite alphabet such that adjacent rows differ in exactly one position and adjacent columns differ in exactly one position. We prove that $m \times n$ Gray arrays exist for $m, n \geqslant 2$ if and only if $|m-n| \leqslant 1$ whenever $\min\{m, n\}$ is even and if and only if $|m-n| \leqslant 2$ otherwise. We construct $n \times n$ and $(n+1) \times n$ Gray arrays for every $n \geqslant 1$ and $m \times (m+2)$ Gray arrays for m odd. A Gray torus is a Gray array such that both the first and last rows are considered adjacent and the first and last columns are considered adjacent. We prove that a Gray array A is a Gray torus if and only if A is a $2m \times 2m$ square array and equivalent to a canonical array.

INTRODUCTION

A Gray code is a listing of all strings of fixed length n over some finite alphabet so that adjacent strings are Hamming distance one apart, i.e., they differ in exactly one entry and the first and last strings in the list also differ in one entry. Equivalently, a Gray code is a Hamiltonian cycle in the n-cube. A classic study of Gray codes and paths and cycles in the n-cube is contained in [1]. Usually the binary alphabet is understood but we consider here an arbitrary alphabet Σ with at least two elements but continue to define adjacency as above. We let σ be the number of elements in Σ.

DEFINITION 1. An array with entries from an alphabet Σ with the property that each row differs in exactly one entry from the rows adjacent to it and each column also differs in exactly one entry from each column adjacent to it is said to be a *Gray array*. Such a matrix with at least two rows and two columns is a *Gray torus* if the last row is adjacent to the first row and the last column is adjacent to the first column.

If $n > 1$ then any $1 \times n$ or $n \times 1$ array with entries $a_1, ..., a_n$ is a Gray array if and only if $a_i \neq a_{i+1}$ for $i = 1, ..., n-1$. Further define an $1 \times n$ or $n \times 1$ array to be a Gray torus if and only if it is a Gray array and, in addition,

0097-3165/92 $3.00

$a_1 \neq a_n$. There are $\sigma(\sigma-1)^{n-1}$ $1 \times n$ Gray arrays over an alphabet with σ elements but only $\sigma(\sigma-1)^{n-2}(\sigma-2)$ $1 \times n$ Gray tori over an alphabet of σ elements. In what follows it will be convenient to omit $1 \times n$ and $n \times 1$ arrays from the discussion.

We remark that while Gray codes as usually defined do not permit repetitions of strings, Definition 1 does not exclude the possibility of repeating rows or columns. Indeed, most of our examples will involve repetitions of rows and columns.

The essentially unique 2×2 Gray array over any alphabet containing at least two distinct symbols a, b is easily seen to be

$$\begin{matrix} a & b \\ b & b. \end{matrix} \tag{1}$$

Note that no 2×2 array with just two a's and two b's can be a Gray array. There do not exist 2×2 Gray arrays containing three distinct elements. We remark that any array equivalent to (1) is also a Gray array in the following sense:

Definition 2. Two Gray arrays are equivalent if one can be obtained from the other by one of the following:

(1) permutation of the underlying alphabet

(2) transposition across the main diagonal

(3) reversal of the ordering of the rows or columns.

Two Gray tori are equivalent if they are equivalent as Gray arrays and/or one can be obtained from the other by cyclic permutations of the rows or the columns.

The array (1) represents a unique equivalence class of 2×2 Gray arrays over any alphabet Σ with $\sigma > 1$. This means that there are exactly $\binom{\sigma}{2}$ equivalence classes of 2×2 matrices determined by distinct pairs of symbols.

To facilitate checking equivalence of Gray arrays we remark that transposition across the back diagonal of an array can be effected by reversing first the order of the rows, then the columns, and finally using ordinary transposition across the main diagonal.

Proposition 1. *Any array equivalent to a Gray array (torus) is a Gray array (torus).*

Gray Arrays from Strings

DEFINITION 3. The *r-windows* of a linear string

$$s = s[1] \cdots s[m] \qquad (r < m)$$

are the $m - r + 1$ substrings:

$$s[i]\, s[i+1] \cdots s[i+r-1], \qquad i = 1, ..., m - r + 1.$$

We denote by $M_r(s)$ the $(m-r+1) \times r$ array of the successive r-windows of a linear string s. That is, the ith r-window of s is the ith row of $M_r(s)$.

Notice that $M_m(s)$ is s and $M_i(s)$ is s viewed as a column.

PROPOSITION 2. *Over any alphabet with at least two elements there exists an $(n+1) \times n$ Gray array for each $n \geqslant 1$.*

Proof. Let a, b be distinct alphabet symbols and consider the array $M_n(s)$, where s is the string $a^n b^n$. Then $M_n(s)$ is a $(n+1) \times n$ Gray array, since s has length $2n$. ∎

PROPOSITION 3. *For each $n \geqslant 1$ there exists an $n \times n$ Gray array.*

Proof. Removing the last row of $M_n(s)$ or, indeed, any Gray array yields a Gray array. ∎

DEFINITION 4. A set of arrays $\{A_k \mid k \in \mathbb{N}\}$ is an embedded family if, for each k, A_k is a subarray of A_{k+1}.

PROPOSITION 4. *For each $n > 1$ there exists an $(n+1) \times n$ Gray array which is part of an embedded family of Gray arrays.*

Proof. The array $M_n(s_1)$ is embedded in $M_{n+1}(s_2)$ as the subarray defined by the last $n+1$ rows and the first n columns where $s_1 = a^n b^n$ and $s_2 = a^{n+1} b^{n+1}$. ∎

PROPOSITION 5. *For each $n > 1$ there exists an $n \times n$ Gray array which is part of an embedded family of Gray arrays.*

Proof. An embedded family is obtained from the proofs of Propositions 3 and 4. ∎

The Dimensions of Gray Arrays

We first examine Gray arrays of lower dimensions. We will see that these cases are essential for the proof of Theorem 1 which determines the possible dimensions of Gray arrays.

LEMMA 1. *If A is any $2 \times n$ $(m \times 2)$ Gray array then $n < 4$ $(m < 4)$. Any 2×3 Gray array is equivalent to one of*

$$\begin{matrix} a & a & b \\ a & b & b \end{matrix} \qquad \begin{matrix} a & a & a \\ a & b & a. \end{matrix} \tag{2}$$

Proof. Suppose A is a $2 \times n$ Gray array. (The argument is similar if A is $m \times 2$.) If the two rows of A have different entries a and b in the first column then they have a common entry, say x, in the second column. Necessarily, x is one of a or b. Otherwise, columns 1 and 2 of A would differ in two entries. Whether x is a or b, A is 2×2 and equivalent to (1). This follows since any third column of A would have the same entry in both rows because the rows of A already disagree in column 1. But then, columns 2 and 3 could not disagree in exactly one entry. The argument is similar if the two rows of A differ in the last column.

Now suppose that the rows of A have differing entries a and b in column j, $1 < j < n$. It now follows from arguments similar to those in the case $j = 1$ that A is of the form

$$\begin{matrix} \cdots & x & a & y & \cdots \\ \cdots & x & b & y & \cdots \end{matrix}$$

for a, b, x, y in Σ with $a \neq b$. But both x and y must be either a or b since the columns of A must differ in exactly one entry. It follows as before that A is 2×3 and it is easily seen that each of the four possibilities are equivalent to one of the two arrays in (2). ∎

LEMMA 2. *If A is any $3 \times n$ $(m \times 3)$ Gray array then $n < 6$ $(m < 6)$. Any 3×5 Gray array is equivalent to*

$$\begin{matrix} b & a & b & b & b & \\ b & b & b & b & b & \cdots \\ b & b & b & c & b & . \end{matrix} \tag{3}$$

Proof. Suppose A is a $3 \times n$ Gray array. (The argument is similar if A is $m \times 3$.) If the first two rows of A have different entries a and b in the first column then they have a common entry, say x, in the second column. Necessarily, x is one of a or b. Otherwise, columns 1 and 2 of A would differ in two entries. If x is b then the first and second entries of the third row must both be b, otherwise, rows 2 and 3 would differ in two entries. Similarly any column 3 must have first and second entries b. Thus A has the form

$$\begin{matrix} a & b & b & \\ b & b & b & \cdots, \\ b & b & c & \end{matrix}$$

where $c \neq b$. Because rows 1 and 2 as well as rows 2 and 3 already differ in an entry, any fourth column of A must have all entries equal to b. Thus A is

$$\begin{matrix} a & b & b & b \\ b & b & b & b \\ b & b & c & b \end{matrix}$$

and no fifth column is possible. Therefore, $n = 4$ in this case.

Now suppose that the first and second rows of A have differing entries a and b in column j, $1 < j < n$. It now follows from arguments similar to those in the case $j = 1$ that A is of the form

$$\begin{matrix} & x & a & y & \\ \cdots & x & b & y & \cdots \\ & z & z & z & \end{matrix}$$

for a, b, x, y, z in Σ with $a \neq b$. But it also follows as before that both x and y must be either a or b since the columns of A must differ in exactly one entry. Therefore, any $3 \times n$ Gray array A must contain one of the blocks:

$$\begin{matrix} & a & a & a & \\ \cdots & a & b & a & \cdots \\ & z & z & z & \end{matrix} \qquad \begin{matrix} & a & a & b & \\ \cdots & a & b & b & \cdots \\ & z & z & z & \end{matrix} \qquad \begin{matrix} & b & a & a & \\ \cdots & b & b & a & \cdots \\ & z & z & z & \end{matrix} \qquad \begin{matrix} & b & a & b & \\ \cdots & b & b & b & \cdots \\ & z & z & z & \end{matrix} \quad . \tag{4}$$

Case 1. The first block cannot be extended in either direction. This follows since necessarily $z = a$ because columns 1 and 2 already disagree in the second position as do columns 2 and 3. Therefore, $n = 3$ in this case.

Case 2. Since columns 2 and 3 differ in the first entry, necessarily $z = b$ in the second block of (4). We cannot extend this block on the right, since every entry of the third column is b. However, on the left a 3×1 column can be added with every entry necessarily a and no further extension is possible. Therefore, $n = 4$ in this case.

Case 3. Similar arguments to those in Case 2 show that the third block in (4) can be extended to the right by one column. Therefore, $n = 4$ in this case as well.

Case 4. The final block of (4) cannot be extended to the left since every entry of column 1 is b. Arguing as before, we see that this block has the form

$$\begin{matrix} b & a & b & b & \\ b & b & b & b & \cdots \\ b & b & b & c & \end{matrix}$$

for some $c \neq b$. A fifth column in this block is possible but must have all entries b. We see that no further extension is possible and, therefore, $n = 5$ in this case. The argument shows further that any 3×5 Gray array is equivalent to (3). ∎

We note that the proof has furnished examples of $3 \times n$ Gray arrays for $n = 3, 4$, and 5.

THEOREM 1. *Let m and n be integers such that $2 \leqslant m, n$. If $\min\{m, n\}$ is even then an $m \times n$ Gray array exists if and only if $|m - n| \leqslant 1$. If $\min\{m, n\}$ is odd then an $m \times n$ Gray array exists if and only if $|m - n| \leqslant 2$.*

Proof. Let A be a $m \times n$ Gray array. We argue by contradiction. Suppose without loss of generality that $m > n + 1$. A has $m - 1$ pairs of adjacent rows and each pair of rows differ in exactly one of n entries. By the pigeonhole principle, there are at least two pairs of adjacent rows of A which differ in the same entry. Suppose $i < j$ and the pairs of rows $(i, i+1)$, $(j, j+1)$ differ in column k. Let row i have kth entry a and row $i+1$ have kth entry b where $a \neq b$.

Since rows i and $i+1$ already differ in the kth entry, rows i and $i+1$ have the same entry in column $k+1$, say y. Either $y = a$ or $y = b$. Otherwise, columns k and $k+1$ would differ in both entry i and entry $i+1$. (If $k = n$ then apply this argument and the following ones to columns k and $k-1$ instead.) Whether $y = a$ or b, columns k and $k+1$ (or $k-1$) differ in either row i or $i+1$ and agree in all other entries.

If $y = b$ then, in particular, row $i+2$ must also have kth and $(k+1)$th entries equal to b. Otherwise, row $i+1$ and $i+2$ would differ in both the kth and the $(k+1)$th entry. Since $i < j$, $i+2 \leqslant j+1 \leqslant m$, and so this argument is not vacuous. Repeating the argument as necessary with succeeding rows, we see that the entries in columns k and $k+1$ of rows $i+1$ through m have exactly the same entry b. This contradicts the assumption that rows j and $j+1$ also differ in column k. Therefore, $y = a$.

Since columns k and $k+1$ are now seen to differ in the $(i+1)$th entry, all other entries in these two columns must be equal. Row $i-1$ (and hence all preceding rows) must have kth and $(k+1)$th entry a. Otherwise, rows $i-1$ and i would differ in two entries. Let row $i+2$ have entry v in both column k and $k+1$. Either $v = a$ or $v = b$. Otherwise, rows $i+1$ and $i+2$ would differ in both columns. (Note that $1 + 2 \leqslant j + 1 \leqslant m$.) Any $(i+3)$th row or subsequent row must also have entry v in columns k and $k+1$. Otherwise, two adjacent rows would differ in both the kth and $(k+1)$th entry.

Applying similar arguments to rows $1, ..., i-1$ and columns $1, ..., k-1$ as necessary, we find that there are also $x, v \in \Sigma$ such that A has the entries indicated in (5).

$$\begin{array}{lcccccc}
 & \text{cols} & k & k+1 & & & \\
 & & a & a & & & \\
 & B & \vdots & & & C & \\
 & & a & a & & & \\
\text{row } i & x\cdots x & a & a & & \cdots & a \\
\text{row } i+1 & x\cdots x & b & a & & \cdots & a \\
 & & v & v & & & \\
 & D & \vdots & & & E & \\
 & & v & v & & &
\end{array} \tag{5}$$

The $(m-2)\times(n-2)$ array $\begin{smallmatrix} B & C \\ D & E \end{smallmatrix}$ obtained by omitting rows i, $i+1$ and columns k, $k+1$ from A is a Gray array. This follows since row $i+1$ and row $i+2$ differ in the kth entry and agree in the entries $1, ..., k-1$, $k+2, ..., n$. But row $i-1$ and row i agree in entries k and $k+1$ and so must differ in exactly one of the entries $1, ..., k-1$, $k+2, ..., n$. Therefore, row $i-1$ and row $i+2$ differ in exactly one entry. There is a similar argument for columns $k-1$ and $k+2$.

When $\min\{m, n\}$ is even, repetition of the above argument leads to a $2\times n'$ or an $m'\times 2$ Gray array for integers m' or $n' \geqslant 2$. The argument can be repeated since the appeal to the pigeonhole principle can always be applied to the resulting $(m-2)\times(n-2)$ array. Suppose the result is a $2\times n'$ Gray array. By Lemma 1, $n' < 4$. Therefore, $|m-n| = |2-n'| \leqslant 1$. The conclusion is the same if an $m'\times 2$ Gray array results.

Conversely, if $|m-n| \leqslant 1$ then Propositions 2 and 3 provide examples of Gray arrays with $|m-n| = 1$ and $m = n$, respectively.

When $\min\{m, n\}$ is odd, the above reduction leads to a $3\times n'$ or an $m'\times 3$ Gray array for integers m' or $n' \geqslant 3$. By Lemma 2, $n' < 6$ or $m' < 6$. Therefore, $|m-n| = |2-n'| \leqslant 2$ or $|m-n| = |m-3| \leqslant 2$.

Conversely, Propositions 2 and 3 provide examples of Gray arrays with $|m-n| \leqslant 1$. If $|m-n| = 2$ then the following construction yields a $(2r+1)\times(2r+3)$ Gray array for each $r > 1$.

$$\begin{array}{ccccccccc}
b & a_1 & b & b & & \cdots & & b & b \\
b & b & b & b & & \cdots & & b & b \\
b & b & b & a_2 & & \cdots & & b & b \\
 & & & & \ddots & & & & \\
 & & & & & b & a_{r+1} & b & b
\end{array} \tag{6}$$

where $a_i \neq b$, $i = 1, ..., r+1$. In (6) the odd numbered columns have constant entry b and the even numbered columns are $[b, ..., a_i, ..., b]^{\mathrm{T}}$ with a_i in position $2i-1$. ∎

We note that (6) can be completed to a Gray array whose dimensions differ by 1 by adding a final constant row of b's.

GRAY TORI

DEFINITION 3. Let a be an arbitrary element of Σ. A *central Gray array* of order m based on a is a square Gray array equivalent to a $2m \times 2m$ Gray array of the form

$$\begin{array}{ccccc} b_1 & a & a & a \cdots & a \\ a & a & a & a \cdots & a \\ a & a & b_2 & a \cdots & a \\ & & & \ddots & \\ & & & b_m & a \\ a & a & a \cdots & a & a \end{array} \tag{7}$$

where $b_i \neq a$, $i = 1, ..., m$.

Note that (1) is central of order 1. Further, note that any array equivalent to (7) is also a Gray torus.

THEOREM 2. *A Gray array A is a Gray torus if and only if A is a $2m \times 2m$ square array and A is equivalent to a central Gray torus of order m.*

Proof. Suppose A has n rows. If $n = 2$ then easy arguments show that a Gray array A is a Gray torus if and only if A is equivalent to a central Gray torus of the form (1).

Assume $n > 2$ and A is a Gray torus. By cyclic permutation of the columns of A, if necessary, we may suppose that the first two rows of A differ in the first column. Hence A is equivalent to an array of the form

$$\begin{array}{ll} a & x \cdots \\ b & x \cdots \\ \vdots & \end{array}$$

for some $x \in \Sigma$. Since the first and second columns of A can differ in only one entry, either $x = a$ or $x = b$.

Suppose first that $x = a$. Since $n > 2$, there are $u, v, w \in \Sigma$ such that A has the form

$$\begin{array}{llll} a & a & u & \cdots \\ b & a & u & \cdots \\ v & v & w & \cdots \end{array}$$

If $u \neq a$ then columns 2 and 3 would disagree in two entries. Hence, $u = a$. In the same way, any other entries in rows 1 and 2 are equal to a. Since rows 2 and 3 cannot disagree in two entries, v is one of a or b.

If $v = b$ then it follows that A has the form

$$\begin{array}{lllll} a & a & a & \cdots & a \\ b & a & a & \cdots & a \\ b & b & & & \\ \vdots & & & B & \\ b & b & & & \end{array}$$

for a subarray B. But since $n > 2$ the first and last rows of A would differ in two entries, contradicting the assumption that A was a Gray torus.

Therefore, $v = a$ and it follows that A has the form

$$\begin{array}{lllll} a & a & a & \cdots & a \\ b & a & a & \cdots & a \\ a & a & & & \\ \vdots & & & B & \\ a & a & & & \end{array} \tag{8}$$

for a subarray B. Furthermore, since the first two rows (columns) of A have only a's in columns (rows) 3, ... each column (row) of B differs from adjacent columns (rows) in exactly one entry. We conclude that B is a Gray array.

Since A is assumed to be a Gray torus and rows 2 and 3 in (8) already differ in the first entry, we conclude that the first row of B has each entry equal to a. Similarly, each entry in the last column of B is a, since the first and last columns of A already differ in the second entry. Since all entries in the first row of A are equal to a, the last row of A must have exactly one entry not a in columns 3, ... Thus we see that the first row of B differs from the last row of B in exactly one entry. Similarly, column 1 of B differs from column n in only one entry. Therefore, B is a Gray torus. By induction, B is square and equivalent to a Gray torus (7) of order $(n-2)/2$. Accordingly, A is a central Gray torus of order $n/2$.

If $x = b$ the argument is similar to the above and A must have the form

$$\begin{array}{lllll} a_1 & b & b & \cdots & b \\ b & b & b & \cdots & b \\ b & b & & & \\ \vdots & & & B & \\ b & b & & & \end{array}$$

for some $a_1 \neq b$, where B is an $(n-2) \times (n-2)$ Gray torus of order $(n-2)/2$. In this case A is easily seen to be equivalent to a Gray torus of the form (7) by interchanging a and b.

Since columns 2 and 3 of A must differ in some entry, the element $b_2 \neq a$ in the first 2×2 principal subarray of B is in column 3 of A (column 1 of B). Since rows 2 and 3 must differ in some entry, b_2 is the (3, 3) entry of A (the (1, 1) entry of B). Repeating the argument as necessary, we see that A is a central Gray array (7).

Conversely, every central array (7) and every array equivalent to a central array is clearly a Gray torus by Proposition 1. ▮

ACKNOWLEDGMENTS

The author is indebted to D. Wiedemann for catching an error in an earlier version of this paper. In a personal communication, L. B. Beasley first suggested that only even-order Gray tori could exist.

REFERENCE

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